

# Analysis of Hedging Strategies Using the Black-Scholes Framework

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# 1. Introduction to Options

The purpose of this project is to examine hedging strategies for options. I will assume that the reader is familiar with options basics such as the definition of a put and a call and how to calculate the payoff for an option, and also has an understanding of put-call parity and similar no-arbitrage arguments.

Throughout this project I utilize the Black-Scholes framework for valuing options. This framework is a simplification of the option markets and equity markets, which relies on a specific set of assumptions. After examining the outcomes of hedging in this framework, I alter the Black-Scholes assumption of the equation that governs the change in the asset price as well as the assumption of constant asset volatility. Specifically, this project examines a situation in which an option dealer buys or sells a large volume of options and, as a result, the asset price changes in a semi-deterministic manner in the time immediately following the trade.

## 1.1 Life as an Options Trader

Individuals and firms that make a living buying and selling options are forced to take on risk in order to do business. This risk arises because by entering into an options contract one is forced into a world where his/her financial future is uncertain. For example, what happens when one buys a European call option contract? The buyer has left a world where he/she invests the call option premium and receives some "sure" risk-free return over the course of the option's life. The buyer enters into a world where he/she loses that call option premium amount today for some payoff in the future that depends on the future price of the asset.

If one wants to make a living buying and selling these contracts (and do so for an extended period of time), it becomes essential to have some ability to minimize the risk. This situation is similar to what is known as the "gambler's ruin" problem<sup>1</sup>. A modified version of the problem is: A gambler starts off with \$10, flips a coin, and every time the coin comes up heads the gambler loses \$1, while every time the coin comes up tails the gambler wins \$1. The gambler continues to play this game until he/she goes broke or has \$20. A natural question to ask is: What is the probability that one will go broke before one reaches \$20?

To analyze the "gambler's ruin" problem, take the general situation where one lets  $C_0$  equal the gambler's cash position at the beginning and  $a > C_0$  is the upper value the gambler stops at if/when  $a$  is reached. Let  $p$  be the probability that the gambler wins \$1, and  $q=1-p$  is the probability that the gambler loses \$1 on any toss. Then, notice that the probability that one will go broke given that one has  $C$  at any time is

$$(1) P(\text{ruin} | C) = r_C = p \cdot r_{C+1} + q \cdot r_{C-1}.$$

Equation (1) follows from the fact that at any time the gambler's probability of being ruined depends only on the current cash position and does not depend on the path that the gambler took

to get to that cash position. This property is known generally as the Markov property. Equation (1) holds for values of  $C \neq a$  and  $C \neq 0$ . At these values of  $C$  the following boundary conditions apply:  $r_0 = 1$  and  $r_a = 0$ . Equation (1) is a recurrence relation that has the following two solutions (assuming  $p \neq q$ ):

$$(2) r_C = 1,$$

$$(3) r_C = \left(\frac{q}{p}\right)^C.$$

These two solutions can be easily verified using substitution. Using the boundary conditions and taking a linear combination of the two solutions, the problem has the following solution:

$$r_0 = A + B\left(\frac{q}{p}\right)^0 = 1 \Rightarrow A = 1 - B$$

$$r_a = A + B\left(\frac{q}{p}\right)^a = 0 \Rightarrow A = -B\left(\frac{q}{p}\right)^a$$

$$B = \frac{1}{1 - \left(\frac{q}{p}\right)^a}, \quad A = 1 - \frac{1}{1 - \left(\frac{q}{p}\right)^a}$$

$$(4) r_C = A + B\left(\frac{q}{p}\right)^C = 1 - \frac{\left(\frac{q}{p}\right)^C - 1}{\left(\frac{q}{p}\right)^a - 1}.$$

Now, when  $p = q = \frac{1}{2}$ , the recurrence relation becomes

$$(5) P(\text{ruin} | C) = r_C = \frac{1}{2} \cdot r_{C+1} + \frac{1}{2} \cdot r_{C-1},$$

and the solutions to this equation are:

$$(6) r_C = 1,$$

$$(7) r_C = C.$$

Again, taking a linear combination of these solutions and using the boundary conditions, the solution is:

$$r_0 = A + B \cdot 0 = 1 \Rightarrow A = 1$$

$$r_a = A + B \cdot a = 0 \Rightarrow A = -B \cdot a$$

$$B = -\frac{1}{a}, \quad A = 1$$

$$(8) r_C = A + B \cdot C = 1 - \frac{1}{a} \cdot C.$$

If an options dealer's goal is to make as much money as possible, he/she will want to set  $a$  to a large value so that as long as one does not go broke one will keep making money. In other words, one is interested in analyzing this game when  $a \rightarrow \infty$ . If  $p \neq q$ ,

$$(9) \lim_{a \rightarrow \infty} r_C = 1 - \frac{\left(\frac{q}{p}\right)^c - 1}{\left(\frac{q}{p}\right)^a - 1} = \begin{cases} 1; & \frac{q}{p} > 1 \\ \left(\frac{q}{p}\right)^c; & \frac{q}{p} < 1 \end{cases}$$

and when  $p = q = \frac{1}{2}$ ,

$$(10) \lim_{a \rightarrow \infty} r_C = 1 - \frac{1}{a} \cdot C = 1; \quad \frac{q}{p} = 1.$$

Therefore, with probability 1 the gambler will go broke eventually regardless of how much money he/she starts out with assuming  $p < q$ . What happens when instead of betting \$1 on each coin toss, the gambler bets some proportion  $1/S$  of his/her initial wealth (where

$S = \frac{C_0}{\text{bet per toss}}$ )? Assuming that  $p > q$  and letting  $a \rightarrow \infty$ ,

$$(11) r_S = \left(\frac{q}{p}\right)^S.$$

As  $s \rightarrow \infty$ , the probability of ruin  $r_S$  tends to 0. The interpretation of this result is that if the probability of winning on each toss is greater than the probability of losing and one has an infinite initial wealth to bet, then one will be able to survive this game indefinitely with a

positive probability. What happens to the probability of ruin  $r_s$  for different values of  $p, q$  and  $S$ ?

## 1.2 Analysis of Risk

The purpose of introducing the “gambler’s ruin” is to show how one can manage risk as a function of the amount that one wagers. From the analysis above we know that when a relatively large amount of one’s total wealth is being wagered, one has a higher chance of going broke at any point. Thus, if one wants to be in the business of buying and selling options (i.e., flipping coins and wagering on the outcomes), then it is best to make smaller wagers rather than large ones. Of course, if one is in the options business, hopefully it is because of some advantage that will help one make a profit (i.e., flipping a biased coin that is weighted in one’s favor). Here are just a few reasons why the odds might be tipped in an option dealer’s favor:

- The dealer has access to private information that will impact the price of options once it is made public.
- The dealer controls fast technology that allows you to quote updated prices on an exchange one step ahead of your competitors.
- The dealer is experienced and employs smart traders with superior decision making skills.
- The dealer charges a commission to act as a counterparty in an option contract with a customer.
- The dealer buys options for a price below the "true" value and sells options for a price above the "true" value. In other words, he/she collects the bid/ask spread by being a market maker.
- The dealer has the ability to consistently price options at their true theoretical value.
- The dealer has the ability to take on a greater amount of risk since he/she benefits from economies of scale of trading a large volume of options.

So, what happens when one has the odds tipped in his/her favor? Now, one has a reason to take a little risk, since one will be getting paid for it more often than not. Take the extreme example: there is a probability of one of winning on each toss; of course, one will bet as much as possible. When one has an edge, there is a tradeoff between taking risk to maximize wealth and reducing risk to ensure one will be able to play the game for a long time into the future. Here is an analysis of the “gambler’s ruin” when the gambler has a large edge versus when he/she does not (note that these probabilities are survival probabilities or one minus the probability of ruin):

**Probability of Survival With and Without Edge ( $a = \infty$ )**

$S=C_0/\text{bet per toss}$	$p = .75$	$p = 0.51$
1	66.667%	3.922%
5	99.588%	18.129%
10	99.998%	32.972%
20	99.999%	55.072%
30	100%	69.885%
40	100%	79.815%
50	100%	86.470%
100	100%	98.169%

**Figure 1.1**

As you can see, having an edge drastically improves the chances of staying in the game indefinitely. But, as  $S$ , the proportion of the total initial wealth bet on each toss, increases, the probability of survival depends less on the amount of edge one has. Once one has an edge and knows that there is a very small chance of going broke, one might be inclined to bet a larger amount of his/her total wealth on each coin flip. In other words, because one enjoys having more money now rather than later one will take some risk by betting a larger amount and hoping to get more money with fewer coin flips.

**Probability of Survival With Edge ( $p = .55$ ) and Varying the Bet ( $a = \infty$ )**

$C_0$	Win/lose \$1 per flip	Win/lose \$5 per flip
5	63.335%	18.182%
10	86.557%	33.058%
15	95.071%	45.229%
20	98.193%	55.187%
25	99.337%	63.335%
50	99.996%	86.557%
100	99.999%	98.193%
200	100%	99.967%

**Figure 1.2**

When the bet is a larger amount of the initial wealth, the probability of survival will, of course, decrease. The reason that one is willing to take more risk is that one will be able to reap the rewards of a larger payoff in the future, relatively sooner.

Essentially, playing the “gambler’s ruin” game is what option traders do for a living. Traders analyze the risk, estimate the edge they have on any single trade and try to put themselves in a position to make a profit. Of course, how much risk one takes on is certainly a matter of preference. With the knowledge of what risk is and why it is important to keep track of

when trading options, I will introduce the risks that come with trading options and what one can do to minimize them.

## 2. Valuing Options Using the Black-Scholes Framework

In the 1973 issue of the *Journal of Political Economy*, Fischer Black and Myron Scholes published a seminal research paper on the pricing of options, “The Pricing of Options and Corporate Liabilities.” In this paper, Black and Scholes derived a closed form solution for the price of an option. They concluded that the value of an option could be determined as a function of time and the price of the underlying asset as well as the parameters: risk-free rate, volatility of the underlying asset, time to expiration, and strike price of the option. In order to arrive at their solution, Black and Scholes laid out some assumptions that I will refer to as the Black-Scholes Framework. Their assumptions are:<sup>ii</sup>

- The risk-free interest rate and the volatility of the underlying asset are known and stay constant through time.
- The stock price follows a random walk in continuous time, and the distribution of stock prices is log-normal.
- The stock does not pay any dividends during the life of the option.
- The option is European style, meaning it can only be exercised at the maturity date.
- There are no transactions costs involved in buying or selling assets or options.
- Assets are divisible, in other words you are allowed to buy a fraction of an asset.
- Short selling is allowed, and there are no restrictions on short selling.

When all of these conditions are satisfied, one can price an option using the Black-Scholes formula, in particular for a European style call:<sup>iii</sup>

$$(12) C(S,t) = S \cdot N(d_1) - E \cdot e^{-r(T-t)} \cdot N(d_2),$$

where:

$$(13) d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}},$$

$$(14) d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}},$$

where  $C$  is the price of a European call option,  $S$  is the underlying asset price,  $N(\cdot)$  is the standard normal distribution function,  $E$  is the strike price of the option,  $t$  is the current time,  $T$  is the expiration time of the option,  $r$  is the risk-free rate, and  $\sigma$  is the volatility of the underlying asset.

Invoking the assumption of no arbitrage, one can use the put-call parity relation,

$$(15) C - P = S - E \cdot e^{-r(T-t)}, \text{ iv}$$

to arrive at the price of a European put option:

$$(16) P(S,t) = -S \cdot N(-d_1) + E \cdot e^{-r(T-t)} \cdot N(-d_2). \text{ v}$$

Throughout this project I used these formulas exclusively to value options in the hedging simulations.

## 2.1 Block Orders and the Informational Role of Trades

Elementary economic theory tells us that the market price for a good is determined by the structure of supply and demand for that good. Is this relationship true as well when it comes to stocks and options? In terms of the micro market structure, on the level of individual orders, the answer is partly yes; the price for an option or stock is influenced by the supply and demand, or liquidity, of the asset. Roughly speaking, liquidity refers to the ease or difficulty with which one can buy or sell an asset in a market. For example, if I want to sell one million shares of company XYZ in the market, I must find a counterparty (or counterparties as the case may be) who will buy these shares from me. In general, when I have fewer shares to buy or sell, I will have an easier time finding a counterparty. An order to buy or sell a *large* number of shares or contracts is known as a block order.

When investors have a block order of stock to trade, they will usually have a stock broker handle the order. The broker will break the trade up into smaller units and buy or sell the stock throughout the course of a trading day, perhaps through different market venues or exchanges. By breaking the trade up into small parts the goal is to minimize the market impact cost - the impact that each trade has on the price of a stock due to the current supply and demand characteristics of that stock.

## 2.2 Placing the Black-Scholes Framework Into A Market

One assumption that the Black-Scholes Framework does not explicitly include, but must be included implicitly, is that the market has infinite liquidity. In other words, if I want to buy or sell an asset at any time, there will always be a counterparty who will allow me to make the trade. This assumption is required in order for the hedging strategy, which involves buying and selling shares of the underlying asset, to work.

When you place the Black-Scholes Framework into a real options market, the assumptions undoubtedly begin to be violated. In particular, the liquidity of an option has a profound impact on how it is priced in a real market. In options markets, this liquidity comes with a cost. For instance, an options market maker will sell ten at-the-money options for less than he will sell one thousand at-the-money options. Obviously, one thousand contracts has

more risk, and if the market maker is wrong on his price he'll have a much larger loss with the larger order size.

## 2.3 Measures of Risk

The analysis of the “gambler’s ruin” game hopefully motivates the need for an option dealer to minimize his/her risk. With this goal in mind, the natural question to ask becomes: What are the risks associated with buying and selling options? In the Black-Scholes Framework, the asset price  $S$  is a random variable, i.e., a risk. If we own an option, we would want to know how the price of that option changes as the asset price changes. This quantity is known as the option *delta*. For a call, one can calculate this risk as:

$$(17) \Delta_c = \frac{\partial C(S,t)}{\partial S} = N(d_1).^{vi}$$

Note that  $0 \leq \Delta_c \leq 1$  (since delta is a value from the cumulative distribution function of a Normal random variable).

Assume that we own one call option and we want to completely hedge this position so that the value of our portfolio will not change randomly no matter how the asset price changes. If at any time  $t$  we own  $-\Delta_c$  shares of the underlying asset, then what will happen to the value of our portfolio  $\Pi$  over a small change in time? Using the Black Scholes stochastic partial differential equation (PDE) and Ito’s lemma it can be shown that:<sup>vii</sup>

$$(18) dV = (\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}) dt + \sigma S \frac{\partial V}{\partial S} dX ,$$

where  $V$  is the value of the option,  $S$  is the asset price,  $\sigma$  is the volatility (or standard deviation of returns of the asset),  $\mu$  is a deterministic average growth rate of the asset,  $dt$  is an infinitesimal change in time, and  $dX$  is a Normal random variable with zero mean and variance

$dt$ . If  $\Pi = V - \frac{\partial V}{\partial S} S$ , then:

$$d\Pi = (\mu S \frac{\partial \Pi}{\partial S} + \frac{\partial \Pi}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2}) dt + \sigma S \frac{\partial \Pi}{\partial S} dX ,$$

$$d\Pi = (\mu S (\frac{\partial V}{\partial S} - \frac{\partial V}{\partial S}) + \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}) dt + \sigma S (\frac{\partial V}{\partial S} - \frac{\partial V}{\partial S}) dX ,$$

$$(19) d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}) dt .$$

Therefore, the change in the value of the portfolio depends only on time and has no stochastic element. In other words, there is no uncertainty.

The argument above motivates a hedging strategy known as delta hedging. Essentially, the purpose of delta hedging is to eliminate the risk of the underlying asset price changing associated with the value of an option. In fact, the delta hedging concept is a key to deriving the closed form solution for the Black-Scholes equation, since now one has the ability to work with a PDE rather than a stochastic PDE. Unfortunately, delta is a variable of time, and in order to maintain a completely delta-hedged portfolio, one must rebalance the portfolio by making sure that at every infinitesimal time step  $dt$ , the portfolio contains delta shares of the underlying asset. Implementing this strategy in reality is not feasible. Instead, one has to choose when to hedge at discrete time intervals.

If the Black-Scholes assumptions were never violated, delta would be the only risk that one would have to worry about in order to manage option risk. Unfortunately, this is not the case in a real market; there are some other risks that are involved with trading options. Any variable (or parameter) that is incorporated into the price of the option and does not stay fixed over time is a source of risk.

In actuality, the volatility of the underlying asset does not stay constant through time. The change in the option price  $V$  with respect to volatility is known as *vega*,

$$(20) \nu = \frac{\partial V(S,t)}{\partial \sigma}.$$

The risk that comes from the change in interest rate is known as *rho*,

$$(21) \rho = \frac{\partial V(S,t)}{\partial r}.$$

One can also measure the sensitivity of the option price with respect to time with *theta*,

$$(22) \Theta = -\frac{\partial V(S,t)}{\partial T}.$$

Finally, the sensitivity of an options delta with respect to the asset price can be calculated to get a second-order measure of how a discrete time delta hedging strategy will perform. This risk is called *gamma*,

$$(23) \Gamma = \frac{\partial^2 V(S,t)}{\partial S^2}.^{viii}$$

### 3. Building a Market Simulation

In order to study delta hedging in a Black-Scholes Framework, it was necessary to create a system that would allow for a simulation of the price of an asset as well as the prices and delta values for a call and put option for that asset. The model was programmed in MatLab and utilized Simulink (a block-diagram based software module used for model-based design) to simulate the model in discrete time.

### 3.1 Implementation of the Black-Scholes Framework

In the Black-Scholes framework the change in the asset price is given by the following stochastic differential equation:

$$(24) dS = \sigma \cdot S \cdot dX + \mu \cdot S \cdot dt, \text{ix}$$

where  $S$  is the asset price,  $\sigma$  is the volatility (or standard deviation of returns of the asset),  $\mu$  is a deterministic average growth rate of the asset,  $dt$  is an infinitesimal change in time, and  $dX$  is a Normal random variable with zero mean and variance  $dt$ . First, it was necessary to implement this model for the asset price to generate a (random) sample path of the asset price. **Figure 3.1** below shows three sample paths for an asset that starts at an initial price  $S_0$  of \$50 and has a volatility of 0.04 with different time steps  $dt$ .

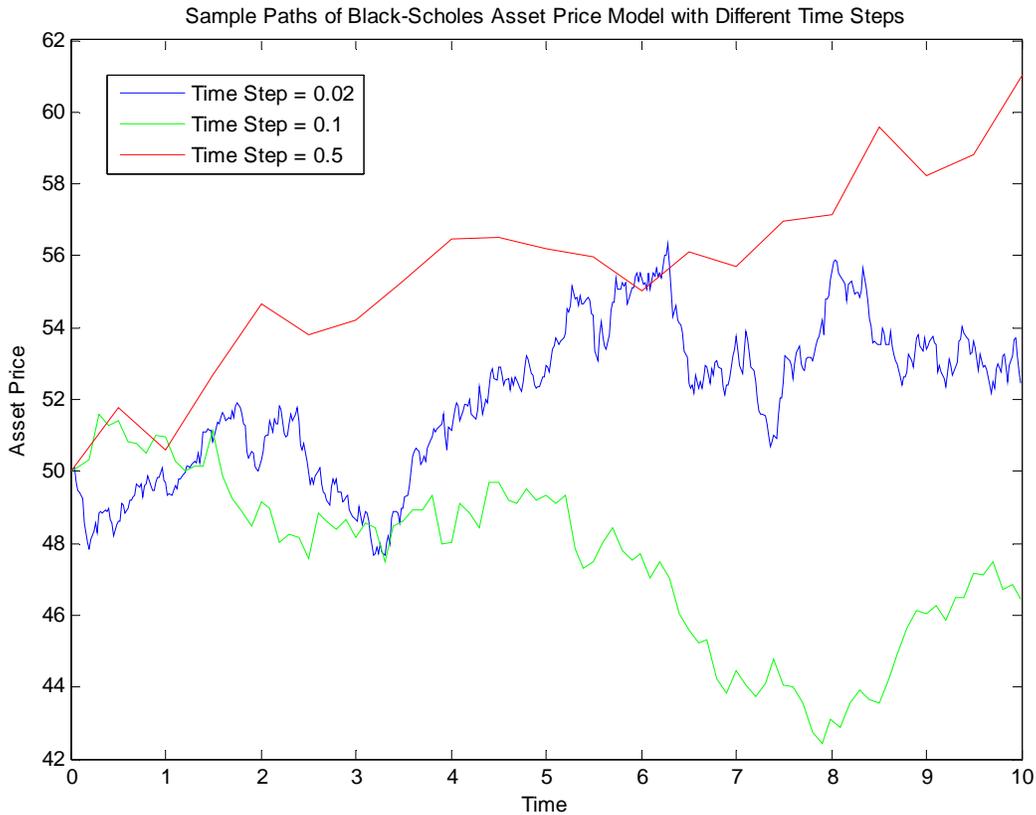


Figure 3.1

The sample paths in **Figure 3.1** appear continuous because the graph linearly interpolates between the discrete points. According to this model, there will be a greater probability that the “jump” that an asset price takes at any time step will be greater in magnitude as the time step increases. Throughout the project  $\mu$ , the growth rate of the asset price, is equal to zero for simplification.

### 3.2 Building a Matlab/Simulink Model

Once a program to generate a sample path for the asset price was in place, a Simulink simulation was designed to simulate the price of a put and call from time zero until expiration. This model requires inputs such as the asset prices at each time step along with the other necessary parameters of the Black-Scholes equation (including volatility, risk-free rate, time to expiration, and strike price). The simulation outputs the price of the call and put as well as the delta of these options at each time step.

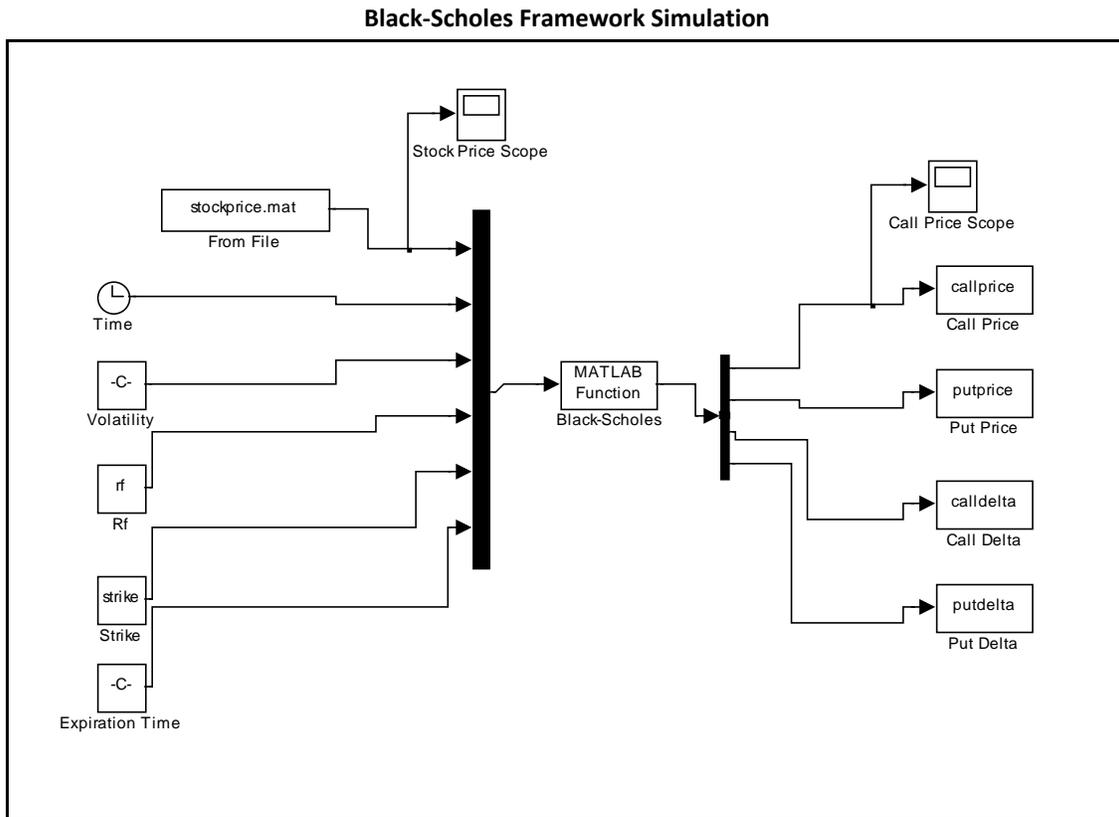


Figure 3.2

After running this simulation, one has the values necessary to examine how a delta hedging strategy would have performed given the particular sample path of the asset price and the other Black-Scholes parameters. It is important to note that in all of the simulations that follow, the Black-Scholes assumption of divisible assets is used very frequently when initiating hedges. As unrealistic as it might seem to buy 0.73849 of a call option, in reality option dealers

trade in large volumes and thus rounding up or down (from 210.73849 options for example) becomes much less relevant during the implementation of a hedging strategy.

### 3.3 Computing the Profit and Loss (p/l) from a Hedging Strategy

The purpose of a hedging strategy is to eliminate risk (and hence, uncertainty). In the Black-Scholes framework, eliminating risk involves buying or selling a certain number of shares of the underlying asset, known as delta hedging. When one creates a truly delta hedged portfolio, one builds a portfolio of assets whose value will not change randomly as a function of time. As mentioned previously, the ability to hedge perfectly (that is to create a portfolio whose value does not change randomly over time) relies on adjusting the portfolio's composition constantly.

Computing the p/l for a portfolio can be a tricky endeavor. At first it seems quite straightforward. One buys an asset at some price, sells it at another price, and then one has either made some money or lost some. However, where did the money to buy the asset come from? Did one have to borrow it (and, consequently, will one have to pay interest on that amount)? Certainly if one sells a stock short one will not take the cash from the sale and let it sit in a non-interest bearing account. Indeed, these arguments are accounted for in the Black-Scholes framework.

I have argued that the delta-hedged portfolio  $\Pi$  is constructed so that it will not change randomly as a function of time. From the analysis of the Black-Scholes PDE and using Ito's lemma one can see that,

$$(25) \quad d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt,^x$$

which is a completely deterministic partial differential equation. Therefore, it follows from the no-arbitrage assumption of the Black-Scholes framework that,

$$(26) \quad d\Pi = r\Pi dt,^{xi}$$

where  $r\Pi dt$  is the amount of interest one earns on an amount  $\Pi$  in an infinitesimal time  $dt$  invested at the risk-free rate  $r$ .

When one delta hedges discretely, equation (25), which dictates the change in the portfolio value, is no longer completely deterministic. Instead, at each discrete time step the change in the value of the portfolio includes a random element that is not hedged away. In particular the Black-Scholes PDE for a portfolio  $\Pi = V - \frac{\partial V}{\partial S} S$  becomes,

$$(27) d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \phi^2 \right) dt, \text{ xii}$$

where it is understood that  $dt$  (which is no longer an infinitesimal time step) is the frequency with which one hedges. The term  $\phi$  is a standard normal random variable. In order to measure the effectiveness of a discrete delta hedging strategy, the p/l equation normalizes the performance of a simulated strategy with what would have been expected under perfect hedging, so that

$$(27) p/l = d\Pi - r\Pi dt .$$

Then, one can easily verify that the expected value of the p/l from the discrete delta hedging strategy will be zero:

$$\begin{aligned} (28) E[p/l] &= E[d\Pi - r\Pi dt] = E\left[\frac{\partial V}{\partial t} dt\right] + E\left[\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \phi^2 dt\right] - r\Pi dt . \\ &= \frac{\partial V}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt \cdot E[\phi^2] - r\Pi dt \\ &= \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt - r\Pi dt \\ &= r\Pi dt - r\Pi dt = 0 . \end{aligned}$$

The final step uses the fact that if a change in a portfolio is completely deterministic then it must return the risk-free rate (or else an arbitrage opportunity would be possible). If one wanted to use this hedging strategy in practice, it would be important to know what the distribution of the p/l looks like given a set of parameters. For example, even if the mean of the distribution is zero, it might have a large variance under certain parameters that might make delta hedging too costly.

## 4. Exploring a Discrete Time Delta Hedging Strategy

In order to test the performance of a discrete-time delta hedging strategy, I modified the model's parameters one at a time and examined how the distribution of the p/l changed as a result. The simulation uses the following base case values (which were chosen partly to reflect a typical set of parameters for an exchange traded option):

<u>Parameter</u>	<u>Base Case Value</u>
Initial Stock Price $S_0$	\$50
Strike Price $E$	\$50
Volatility $\sigma$	$0.2 \sqrt{\text{years}}$
Risk-Free Rate $r$	5.0% per year
Time Step $dt$	0.005 years ~ 1.825 days
Time to Expiration $T$	0.2 years ~ 73 days

**Figure 4.1**

The base case values are adjusted in the simulation one parameter at a time. In total, the system ran simulations with 16 sets of parameters: the base case parameters and then the base case with the following changes made one at a time:

<u>Strike Price</u>	<u>Volatility</u>	<u>Risk-Free Rate</u>	<u>Time Step</u>	<u>Time to Expiration</u>
45	0.1	3.0%	0.0025	0.1
50*	0.2*	5.0%*	0.005*	0.2*
55	0.3	7.0%	0.01	0.5
	0.4		0.05	1.0
	0.5		0.1	

**Figure 4.2**

For each set of parameters, the program looped through the simulation 5,000 times and recorded the final p/l value at the end of each simulation. In order to gain some insightful

information about the distributions, the program computed the mean and variance of each data sample. In order to characterize the distribution, I first noticed with a visual inspection that the base case distribution of p/l values appeared to be normally distributed. With this observation in mind, I chose to use a statistical test that requires an assumption for a specified distribution (in this case, normal) rather than one that is non-parametric (such as the Kolmogorov-Smirnov test). In particular, I chose the Anderson-Darling test because of the ease of implementation and because there are readily available tables of test statistics for this test assuming a specified distribution of normal.

#### 4.1 Test for Normality and the Base Case Results

Using the base case parameters provided above, a simulation with 5,000 iterations resulted in an average p/l of \$0.001286. This result is very close to the analytically determined expected value of zero. The sample variance of the data was 0.057707064. Under the null hypothesis that the data are normally distributed, the Anderson-Darling test statistic  $A^2 = 14.58832$ . At an  $\alpha = 0.05$  level of significance one concludes that this data does not come from a normal distribution since the test statistic is much larger than 0.752, the upper bound of the critical region.

Below is a plot of the p/l distribution compared to a normal distribution with the same mean and variance of the p/l distribution. From a visual inspection it is clear that the p/l distribution has a significantly larger amount of area centered near the mean.

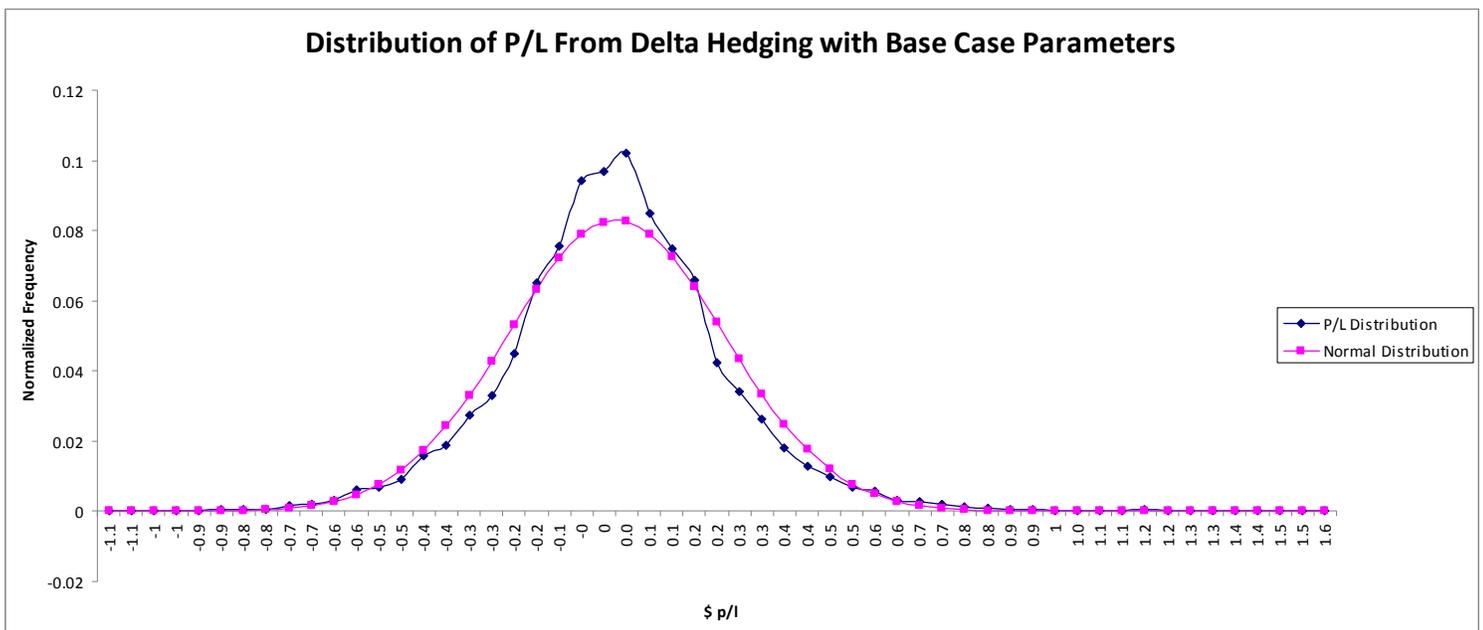


Figure 4.3

The fact that this distribution is tightly centered about the mean is beneficial for a trader implementing a delta hedge for an option portfolio. In the sections that follow, different

parameters and variables of the Black-Scholes framework will be modified one by one to determine the impact on the distribution of p/l.

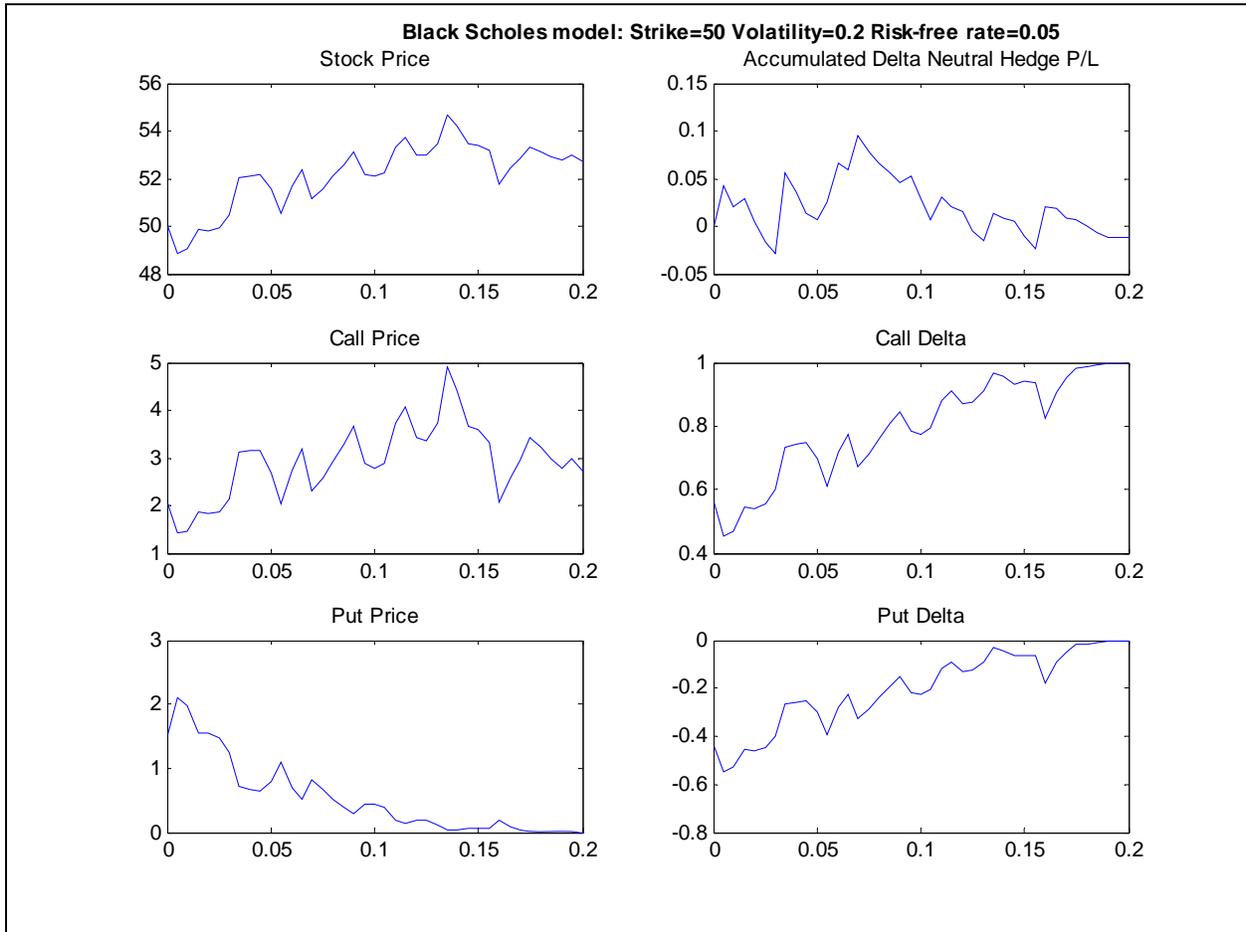


Figure 4.4

Figure 4.4 shows the results from one simulation using the base case parameters. The top left plot contains the discrete values of the asset price from  $t = 0$  to  $t = 0.2$  at time step increments of  $dt = 0.005$ . The top right plot is the accumulated p/l from a delta hedging strategy over the life of the option. The final four plots contain the call price, put price and their respective delta values for an option with a strike price of \$50.

## 4.2 Analyzing the P/L as a Function of the Strike Price

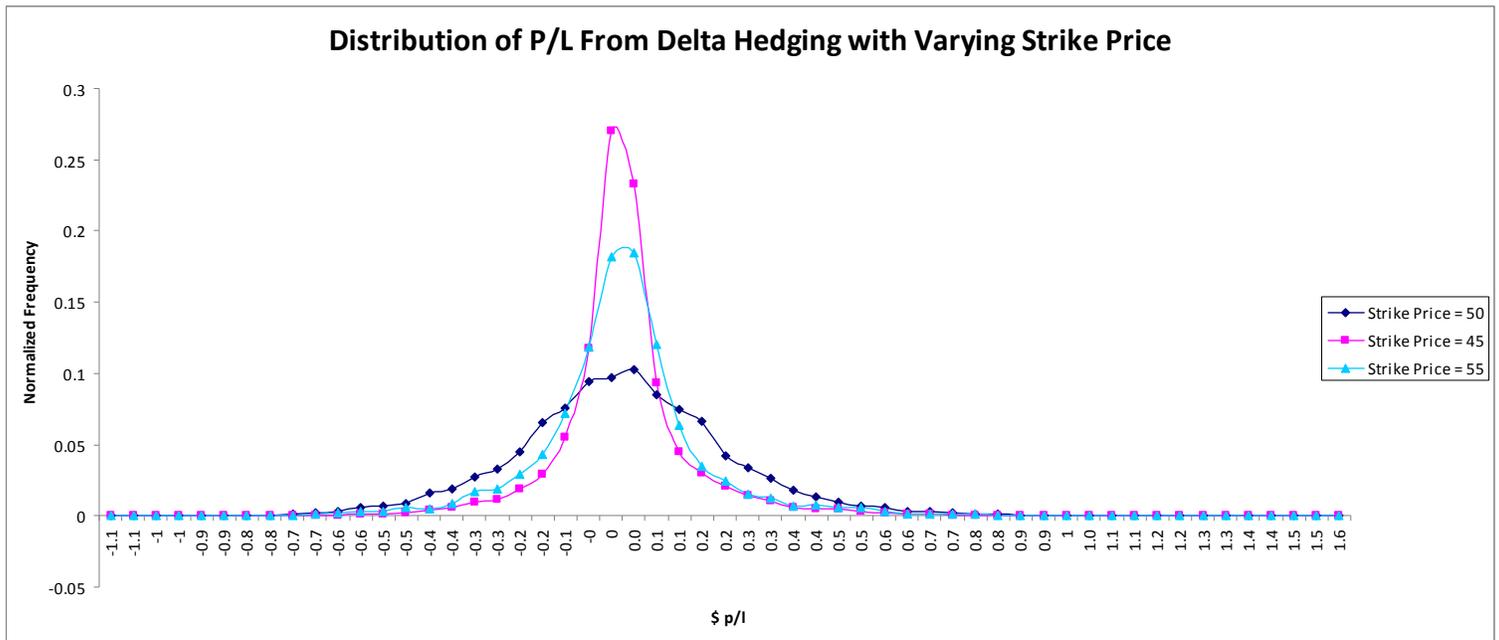
The strike price for the base case is an option that begins at the money (ATM) with a strike price of \$50 and the initial asset price of \$50. What happens to the p/l of the delta hedging strategy when one is forced to hedge an option that is initially either out of the money (OTM) or in the money (ITM)? As one can see from the table below it appears that the variance of the p/l distribution is lowest for an OTM option. The variance for the ITM option is greater than the base case ATM option.

**P/L Distribution Statistics with Adjusted Strike Price**

<u>Base Case w/ Adjusted Strike Price</u>	<u>P/L Average</u>	<u>P/L Variance</u>
45	\$0.003850916	0.021743702
55	-\$0.000769681	0.0317708

**Figure 4.5**

Below is a plot of the distribution for the p/l for each of the strike price parameter values. (Note that the ATM distribution has the same parameters as the base case).



**Figure 4.6**

Based on the simulations it appears that the p/l distribution is centered more closely toward the mean for options that are not ATM. The implication from this result is that a trader who wishes to hedge an option that is ATM will have a harder time keeping his/her p/l flat relative to when the option is not ATM. Intuitively, this result is consistent with what one would expect. An option that is OTM or ITM will have a higher probability of becoming deep OTM or deep ITM than an option that starts out ATM. An option that is deep ITM is easy to hedge with relatively greater precision because the option has a value that moves almost in step with the asset price. Since the option is deep ITM, it will have a delta of about  $\pm 1$  and the delta will not change that much for any small changes in the asset price. Similarly, a deep OTM option has nearly no value (and has a delta of about 0) so it will not change value that much with any small change in the asset price.

### 4.3 Analyzing the P/L as a Function of the Volatility

An asset's volatility is a measure of the uncertainty of the future value of that asset. In particular, an asset with a high volatility will have a greater chance of having a much different price after each time step than an asset with a low volatility. Therefore, one would expect that implementing an imperfect hedging strategy that attempts to minimize the risk from a financial derivative of the asset would be less successful as the future price of the asset becomes more uncertain (i.e., the volatility of the asset increases). The table below which presents the statistical data of the p/l distribution as a function of the asset volatility supports this conclusion:

**P/L Distribution Statistics with Adjusted Volatility**

<u>Base Case w/ Adjusted Volatility</u>	<u>P/L Average</u>	<u>P/L Variance</u>
0.1	-\$0.00079203	0.014946856
0.3	\$0.008570923	0.125180429
0.4	-\$0.005447194	0.22363503
0.5	\$0.002649686	0.353006253

Figure 4.7

It is important to note that in an actual market, volatility is not necessarily synonymous with risk. However, volatility is a component that can drastically affect the riskiness of a portfolio. An option with a relatively high volatility will be tougher to hedge consistently versus an option with a low volatility. Below is a plot of the p/l distributions for the four volatility parameters where the other Black-Scholes parameters are the base case values.

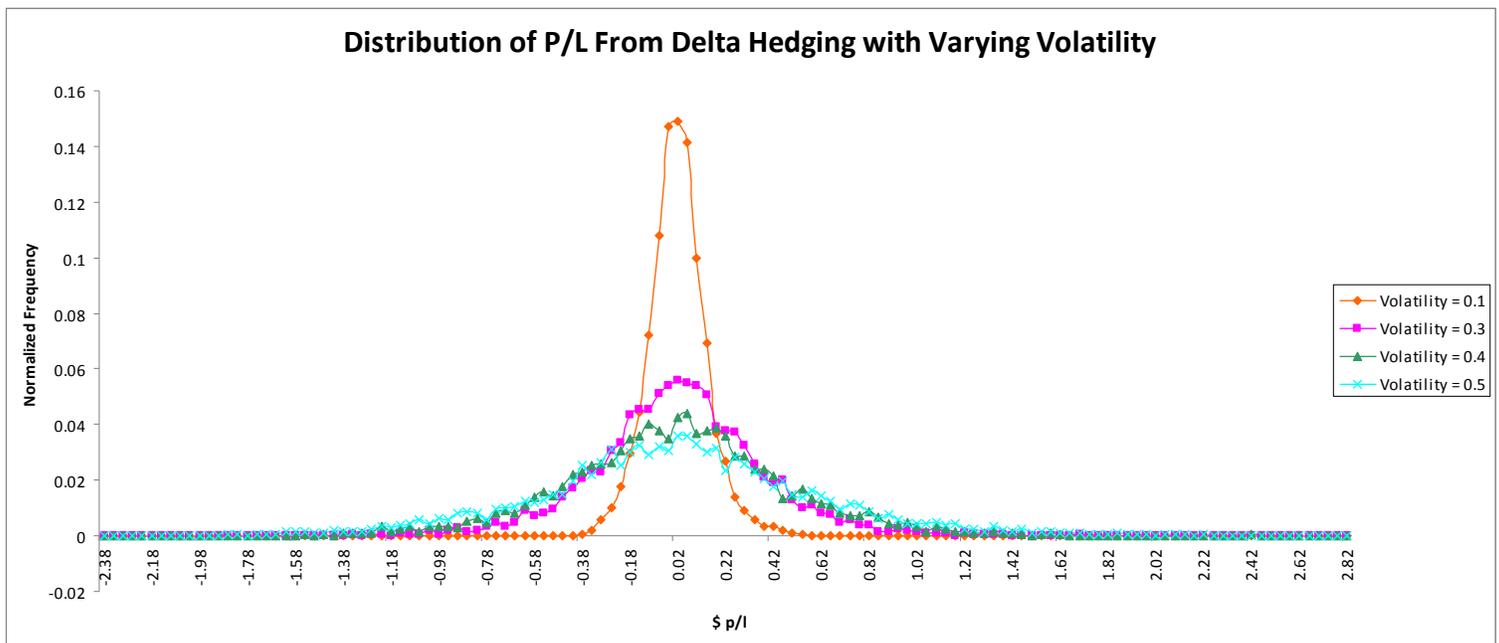


Figure 4.8

#### 4.4 Analyzing the P/L as a Function of the Risk-Free Rate

Another parameter of the Black-Scholes equation that factors into the price of an option is the risk-free rate. Below is a table that contains statistical data of the p/l distribution when the risk-free rate is 3.0% and 7.0%.

<u>Base Case w/ Adjusted Risk-Free Rate</u>	<u>P/L Average</u>	<u>P/L Variance</u>
3.0%	\$0.000779669	0.058520659
7.0%	\$0.001636135	0.05722351

**Figure 4.9**

From the table one can see that varying the risk free rate does not seem to have much of an impact on the distribution of the p/l. In order to have more conclusive evidence that the variance of the p/l will not change as a function of the risk-free rate it would be necessary to explore more simulations in which there is a wider range risk-free rates.

#### 4.5 Analyzing the P/L as a Function of the Time Step

The frequency of hedging is a very important choice that one must make when deciding to hedge a position. In order to hedge perfectly, one would want to be able to hedge as frequently as possible. However, in real markets transaction costs make it suboptimal to hedge very frequently. In order to determine a good hedging frequency one needs to weigh the gains that result from reducing risk against the costs of hedging. In practice, this procedure is often based on a trader's risk tolerance and expectations about the market. Below is a table that shows the statistics of the p/l distribution when the hedging frequency is varied. The variance of the p/l distribution increases drastically as the number of hedges decreases during the life of the option.

<u>Base Case w/ Adjusted Time Step</u>	<u>Number of Discrete Hedges</u>	<u>P/L Average</u>	<u>P/L Variance</u>
0.0025	80	\$0.000238875	0.029264171
0.01	20	\$0.002889287	0.118686331
0.05	4	-\$0.00873964	0.530963737
0.1	2	\$0.031362058	1.003789532

**Figure 4.10**

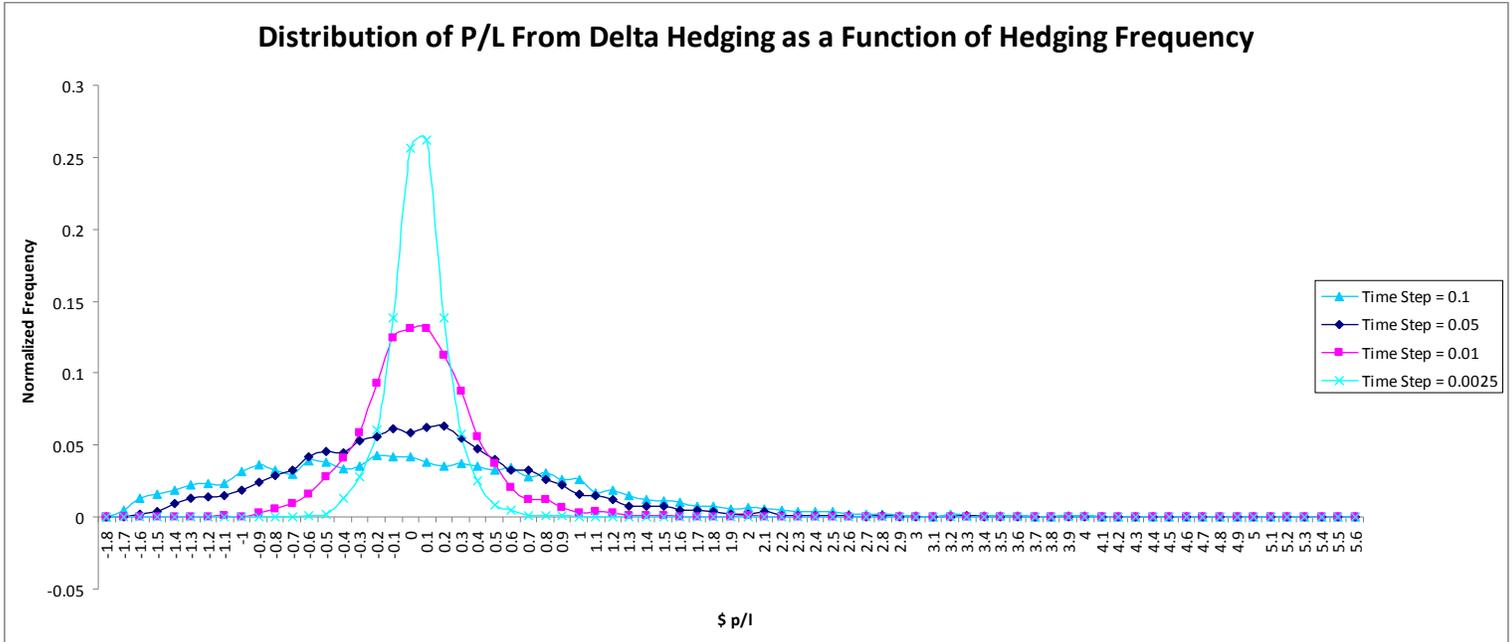


Figure 4.11

As one decreases the number of hedges it becomes apparent that the p/l distribution is a skewed distribution. The right tail of the distribution contains values that are further away from the mean than the left tail. But, because the distribution is not symmetric and the mean is approximately zero, a greater amount of the area of the distribution will be to the left of zero. In general, a trader wants the downside of his/her position to be limited. Therefore, a trader will desire this lack of symmetry in the p/l distribution. When a low probability tail event does occur, the trader will experience a relatively limited downside loss but could gain a large amount on the upside.

#### 4.6 Analyzing the P/L as a Function of the Time to Expiration

The time to expiration of an option is another factor that appears to influence the results of the delta hedging strategy. As the time between when the option position is initiated and expiration increases, the variance of the p/l distribution decreases. Results from simulations where the time to expiration began at 0.1, 0.5 and 1.0 years are presented in **Figure 4.12**. One explanation for this result is that in a situation with a lower time to expiration, there are fewer hedges and thus fewer opportunities for individual hedges to offset each other at each time step.

**P/L Distribution Statistics with Adjusted Time to Expiration**

<u>Base Case w/ Adjusted Time to Expiration</u>	<u>P/L Average</u>	<u>P/L Variance</u>
0.1	\$0.002597826	0.059483222
0.5	\$0.001962607	0.057018045
1.0	-\$0.0065136	0.054395568

**Figure 4.12**

The above analysis has explored how the delta hedging strategy holds up as a discrete hedging strategy under various parameters. The final section of this project investigates a particular situation in which an option dealer hedges a large block order.

## 5. Modeling and Hedging Block Order Option Trades

The analysis of the discrete time delta hedging strategy rests on all of the assumptions that are implicit in the Black-Scholes framework. Unfortunately, in real options markets many of these assumptions either do not hold at all or must be appropriately modified. In particular, volatility of the underlying asset is not a constant in real options markets. This section explores a particular scenario in which a trader is forced to relax the assumption of constant volatility.

Consider an option dealer who executes trades for his/her customers. Sometimes this dealer is faced with a customer who wants to buy or sell an option block order. How is this option dealer (who is forced to take on a large option position) able to hedge this position to minimize his/her risk? In order to study this situation it is first necessary to completely describe the scenario in question.

### 5.1 How an Options Dealer Is Forced into a Large Position

Here is a step by step scenario of why an option dealer might be forced to hedge a large option position:

- An option broker/dealer firm has a trading desk that is equipped to quote option prices for their institutional clients.
- A client calls the desk and wishes to buy 2,000 January 2009 call options with a \$50 strike in stock XYZ.
- The trader sees that the collective ask size offered on all of the electronic option exchanges at the best price is about 300 contracts. The trader quotes an ask price of  $\$W.WW + \$\Delta.\Delta\Delta$  for the full size with the intention of acting as a dealer, rather than attempting to buy these options for the customer in the open market.  $\$W.WW$  is the best current ask price on the open market.  $\$\Delta.\Delta\Delta$  is a premium or mark-up on the current price that reflects the informational component of the trade as well as the large risk that the company must take on to write these options. The trader will write 2,000 contracts and sell these 2,000 contracts to the customer.
- Now the trader has a liability. (Namely, he has the possible obligation to sell 2,000 shares of XYZ to the customer at some future date). The trader wishes to neutralize this liability by hedging away this risk. The trader has to buy some stock, buy call options or sell put options in order to eliminate this risk. The hedging must be recalculated and implemented frequently in order to continuously minimize the risk.

### 5.2 Alternative Methods of Executing a Customer's Option Block Order

A natural question that is avoided by the explanation in the scenario above is: Why would the option dealer wish to sell 2,000 contracts at a price of  $\$W.WW + \$\Delta.\Delta\Delta$  instead of buying the current 300 that are being offered at  $\$W.WW$  and then either buying the remaining 1,700 in

the open market as they become available or writing the remaining 1,700 at a price of  $\$W.WW + \$\Delta.\Delta\Delta$ ? In fact, there are essentially three ways the options dealer can handle this trade. Below is a table that outlines these alternatives and the pros and cons of each.

Alternative Methods of Executing a Customer's Block Order

<u>Trade Alternative</u>	<u>Timeline of trade</u>	<u>Timeline of hedge</u>	<u>Pros</u>	<u>Cons</u>
Write all 2,000 contracts	Trade is executed immediately	Hedge is executed immediately after trade is announced	Able to hedge before "information" contained in the trade is priced into the market	Most difficult alternative to hedge since trader must interpret the effect of the information from the trade
Buy 300 contracts in the market; write the remaining 1,700 contracts	Small part of trade is executed immediately; a short time after this the balance of the order is executed	Hedge is executed after the balance of the order is executed	Buy 300 contracts for the customer at a "low" price	Hedge will be executed at a price that reflects some of the trade's information
Buy 300 contracts; keep buying up contracts offered in the market up to 2,000 total	Small parts of the trade are executed over time as a function of market liquidity	Hedge is executed after the full amount of the customer's trade has been executed	Buy 300+ contracts for the customer at a "low" price	Hedge will be executed at a price that reflects all of the trade's information

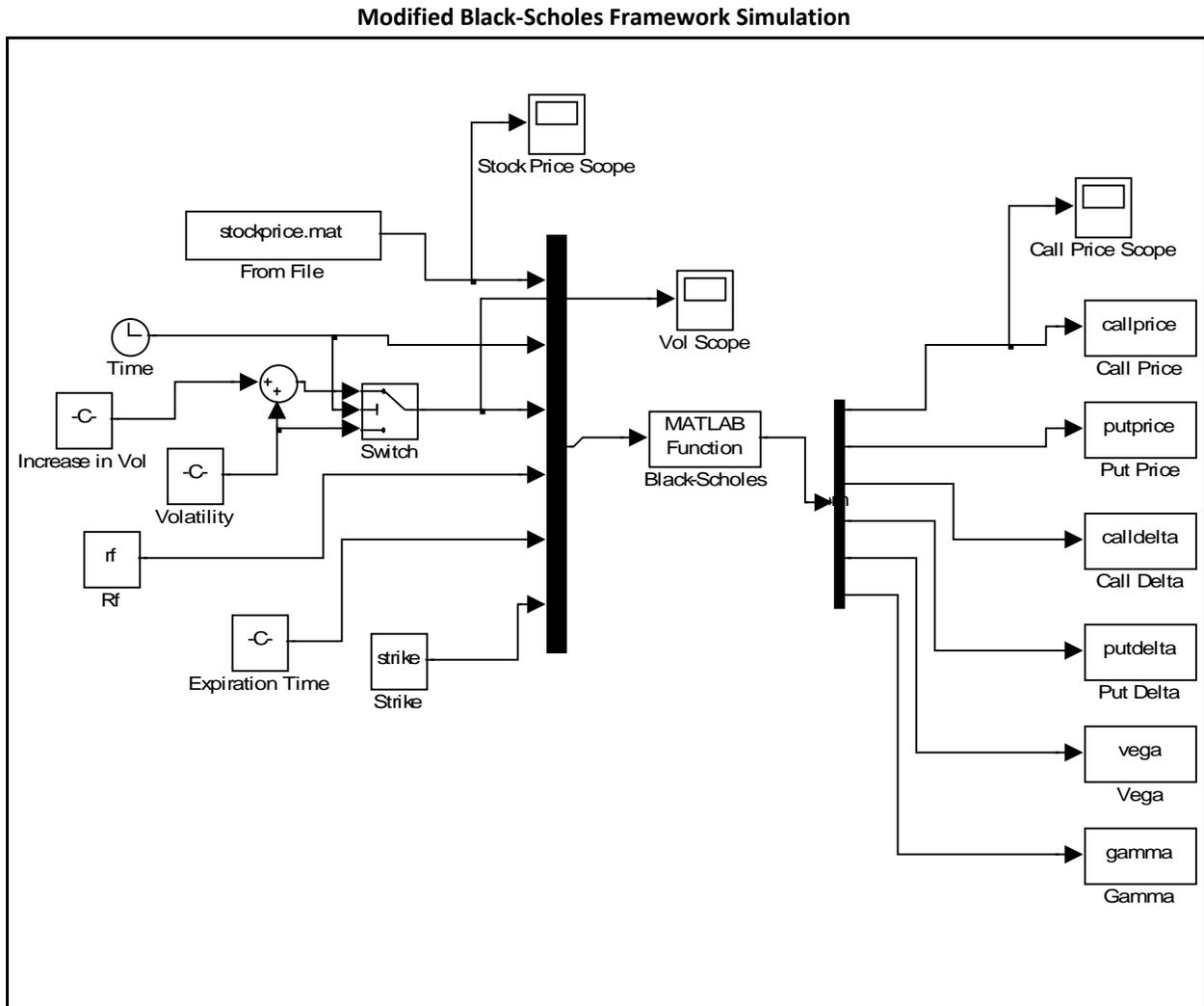
Figure 5.1

The reason that an options dealer would choose the first alternative over the other two is that delaying the hedge is very costly. As the market prices are adjusted based on the customer's order the options dealer will be losing a very valuable opportunity to hedge his/her risk at a "discounted value." This situation arises from the fact that the options dealer has some information that the rest of the market does not have: there will be a large block order in the near future that will alter the price of options on that particular asset (as well as the asset price itself).

### 5.3 Modeling a Block Order with an Informational Impact

In order to model the block order scenario I make a small modification to the Black Scholes Framework. For the time step immediately following the block order trade, time  $t + 1$ , I assume that the asset price  $S_{t+1}$  increases or decreases (with an equal probability) by a random amount  $u$ , which is a uniform random variable on the interval  $[0.05 \cdot S_t, 0.1 \cdot S_t]$ . In other words, I assume that as a result of the customer buying a large number of call options on this asset, the price of the asset will change by somewhere between 5-10% one time step from now. In

addition to this, I also assume that the volatility of the asset increases by a fixed amount:  $vol_{inc.} = 0.1$  at the time step following the trade. **Figure 5.2** is the Simulink block diagram that was used in this section to implement the desired changes in the model.



**Figure 5.2**

The Switch block in the model is programmed to include the increase in volatility when the simulation time exceeds the time of the block order trade. Also, the method that generates the stock price data was adjusted according to the modifications mentioned above. The rationale for this modification to the model is that it represents a scenario in which the customer has some vital information about the asset that will make the asset price more volatile in the near future. This information could be anything from an earnings announcement to a press release about the company being acquired by another company. As the seller of these options, how can you protect yourself by hedging away your risk before the next time step?

Below in **Figure 5.3** are two sample paths of the asset price when the Black Scholes Framework is modified to account for the informational role of this trade. The trade takes place at time  $t = 0.1$ , and the volatility of the asset before this time is 0.2. After the trade occurs the volatility of the asset is 0.3. Notice that in one sample path the asset price jumps up right after the trade, while in the other sample path the asset price drops a significant amount at time 0.1. The jump up or down (which occurs with equal probability) is chosen uniformly between 5 – 10% of the current asset price.



**Figure 5.3**

#### 5.4 Risks Associated with a Block Order

By introducing a change in the volatility of the underlying asset in this scenario it becomes apparent that some risk involved in buying and selling options is a function of the volatility. In fact, volatility is not a constant parameter in actual options markets. Volatility is actually a function of time as well as strike price. Assume that one knows all of the Black-Scholes parameters for an option except the volatility. Then, using the current market price for the option, one can use the Black-Scholes pricing formula to explicitly solve for the market

“implied” volatility. When one solves for the implied volatilities for options with different strike prices and times to maturity (but keeping the same underlying asset) it is possible to construct a volatility surface. A volatility surface is a three dimensional surface that plots the volatility as a function of both time and strike price. Volatility surfaces and implied volatilities are used by option market makers to fine tune their bid and offer prices for options.

The risk that depends on changes in volatility is known as *vega*, which was mentioned previously in section 2.3. By taking the partial derivative with respect to the volatility one finds that,

$$(29) \nu = \frac{\partial V(S,t)}{\partial \sigma} = S \cdot N'(d_1) \cdot \sqrt{T-t}, \text{ xiii}$$

where  $N'(x)$  is the probability density function of the Gaussian random variable.

Since hedging is performed in the simulation at discrete intervals it will also be important to hedge the second-order sensitivity of the option price with respect to the asset price, *gamma*. Gamma can be computed by taking the derivative of delta (for either a call or a put) with respect to the asset price:

$$(30) \Gamma = \frac{\partial^2 V(S,t)}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \frac{\partial N(d_1)}{\partial S} = \frac{N'(d_1)}{\sigma \cdot S \sqrt{T-t}}. \text{ xiv}$$

One should be able to hedge all of the risk associated with the informational impact using delta, gamma and vega appropriately. It is important to note that one must buy or sell options in order to hedge gamma and vega since the underlying asset has no gamma or vega risk.

## 5.5 Determining a Hedging Strategy

The goal of this section is to propose a hedging strategy that will minimize the p/l for a trader that was forced to sell a block order of call options to a customer. The set-up of this scenario is:

1. At a time right before  $t = 0.05$  the option dealer writes a large number of call options for a customer’s order. The dealer is now short call options.
2. The dealer knows that the order will have an immediate impact on the price of the underlying asset (according to the specification in section 5.3) and knows that the volatility of the underlying asset will also jump from its current value of 0.2 to 0.3.
3. Understanding the dynamics of the trade, the dealer calculates the delta, gamma and vega risk of his/her position using the **new volatility value** and hedges immediately before the market adjusts to the new information. The key to this successful hedging strategy is that the trader recognizes that by measuring the risk of his/her position as if

the information had already impacted the market he/she is able to more effectively hedge for the future world that involves a more volatile underlying asset.

4. The dealer then puts in place a hedge for the time step when the volatility jumps by buying and selling a combination of options and the underlying asset. After the volatility increases and the asset price jumps up or down, the asset price will again follow a path determined by the Black-Scholes framework. Therefore, once the asset price starts to follow a “regular” path again, the dealer will just use a simple delta hedging strategy.

The dealer’s goal is to determine the correct number of options and underlying assets to buy or sell at the time step immediately before the information impacts the market. Before I explore how the dealer should go about this task, it is first beneficial to discuss some of the practical and technical aspects of hedging a customer’s order when a dealer acts on behalf of a customer.

## 5.6 Discussion of Front-Running and a Broker’s Obligations to a Customer

When a dealer has agreed to make a trade for a customer, the dealer is not allowed to buy or sell any related assets until the customer’s order has been executed. This rule, which prevents a dealer from “front-running” a customer’s order, is put in place to protect the customer from suffering adverse price movements as a result of the dealer’s knowledge of the trade. Therefore, the dealer must wait until an order is “announced” to a market before he/she can hedge the order.

The fact that option orders are executed both in open-outcry markets as well as electronic markets presents an opportune scenario for a dealer to hedge a block order trade. After executing a block order trade in a **physical open-outcry market** the trade is considered “public information” and the dealer is then free to hedge the order without fear of being in violation of “front-running” rules. But, there is a short, but significant lag time between when an order is made public on an **exchange floor** and when the traders that quote prices on **electronic markets** learn about the trade and incorporate it into their price. As a result of this inefficiency, the dealer has a short window of time to hedge the deal at market prices that **do not** reflect the informational role of the block order trade. This result is the main rationale for why a dealer will want to write all the contracts of the customer’s order rather than buy them over the course of time in the open market. The dealer has an element of surprise, which allows him/her to reduce risk efficiently and cheaply.

## 5.6 Implementing a Vega/Delta/Gamma Hedging Strategy

In the simulation, a dealer seeks to hedge a short position in the call options with a strike price of \$50. The simulation parameters are the same as the base case values used previously except for the fact that now the dealer is short 10 call options and the time step  $dt = 0.0025$ . The trade takes place at  $t=0.05$ . The 10 call options that the dealer has sold are known as the dealer’s *position*. The risk associated with this position is known as the *position risk*.

In the discussion from section 5.5 the dealer was faced with the problem of choosing a hedge for one particular time step (the one immediately after executing the customer's trade). The dealer's problem involves determining how to minimize the position risk by buying and selling other options and the asset. In general exchange traded options for an underlying asset will come in many strike prices and expiration dates. Thus, a trader will have many options to choose from when determining which options to buy or sell in order to hedge the position risk. In actual markets, a trader will take into account the market prices of the options available for hedging in an attempt to exploit any pricing inefficiencies. The dealer will buy options for the hedge that are relatively cheap in the market and sell options that are relatively expensive, while at the same time having the goal of minimizing the position risk. To simplify the situation in the simulation there are call and put options with strike prices of \$45 and \$55 to use for hedging purposes with the same expiration date as the options the dealer needs to hedge.

To calculate the hedge, the dealer determines his/her position risk (using the updated volatility value). Next, the dealer calculates the risks of the options that are available for hedging. The goal is to choose a linear combination of the options and the asset to offset the position risk. This problem is represented by a linear system that can be solved using linear algebra. Unfortunately, the system is underdetermined and, therefore, there is an infinite number of combinations of solutions that will make the individual components of risk equal to zero. The equation of the linear system in matrix form is,

$$(31) \begin{bmatrix} \Delta_{45}^C & \Delta_{55}^C & \Delta_{45}^P & \Delta_{55}^P & 1 \\ \Gamma_{45}^C & \Gamma_{55}^C & \Gamma_{45}^P & \Gamma_{55}^P & 0 \\ \nu_{45}^C & \nu_{55}^C & \nu_{45}^P & \nu_{55}^P & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \Delta_{50}^C \\ \Gamma_{50}^C \\ \nu_{50}^C \end{bmatrix},$$

where  $\Delta_X^w, \Gamma_X^w, \nu_X^w$  are the delta, gamma and vega respectively of an option where  $X$  is the strike price and  $w$  denotes whether it is a call 'C' or put 'P'. The hedge coefficients  $x_i$  represent the number of each type of option the dealer will buy or sell for the hedge (where it is understood that a negative  $x_i$  means the dealer will sell that option or asset). In order to choose a solution for the linear system I modified the problem into the following constrained nonlinear minimization problem:

$$(32) \quad \min \left( \left| \sum_{i=1}^5 V_i x_i \right| + \sum_{i=1}^5 |x_i| \right), \text{ where } V_i \text{ is the value of option or asset } i,$$

$$s.t. \quad \begin{bmatrix} \Delta_{45}^C & \Delta_{55}^C & \Delta_{45}^P & \Delta_{55}^P & 1 \\ \Gamma_{45}^C & \Gamma_{55}^C & \Gamma_{45}^P & \Gamma_{55}^P & 0 \\ \nu_{45}^C & \nu_{55}^C & \nu_{45}^P & \nu_{55}^P & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \Delta_{50}^C \\ \Gamma_{50}^C \\ \nu_{50}^C \end{bmatrix},$$

where  $-10 \leq x_i \leq 10$  for  $i = 1, \dots, 5$ .

The objective function in equation (32) is motivated by two goals. The first part of equation (32) puts emphasis on reducing the cost of the hedge. This term assures that the hedge will not be costly to initiate. The second term in equation (32) attempts to decrease the total number of options and assets bought or sold. In practice, one desires to trade the fewest number of options when hedging because of liquidity constraints. The constraints in this minimization problem ensure that the position risks will be set to zero. With the objective function in place, the simulation used the Matlab command *fmincon*, which attempts to find the minimum of a constrained nonlinear multivariable problem. This function uses a Sequential Quadratic Programming method. This algorithm iterates over sub-problems and solves Quadratic Programs of the sub problems, while incorporating a line search method at each major iteration to form a search direction.

### 5.7 Hedging P/L Results for an Information Impact Trade

A sample result of the minimization problem in equation (32) can be found in **Figure 5.4**.

**Sample Solution for Hedge**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5.5025 \\ 7.3236 \\ 0.0001 \\ -0.0001 \\ -0.8769 \end{bmatrix} = \begin{bmatrix} 45 \text{ Strike Calls} \\ 55 \text{ Strike Calls} \\ 45 \text{ Strike Puts} \\ 55 \text{ Strike Puts} \\ \text{Underlying Asset} \end{bmatrix} \text{ is the solution that minimizes } \left( \left| \sum_{i=1}^5 V_i x_i \right| + \sum_{i=1}^5 |x_i| \right) = 13.7036$$

while satisfying the following set of equations which set the individual risks of the position to zero

$$\begin{bmatrix} \Delta_{45}^C & \Delta_{55}^C & \Delta_{45}^P & \Delta_{55}^P & 1 \\ \Gamma_{45}^C & \Gamma_{55}^C & \Gamma_{45}^P & \Gamma_{55}^P & 0 \\ \nu_{45}^C & \nu_{55}^C & \nu_{45}^P & \nu_{55}^P & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} - \begin{bmatrix} \Delta_{50}^C \\ \Gamma_{50}^C \\ \nu_{50}^C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

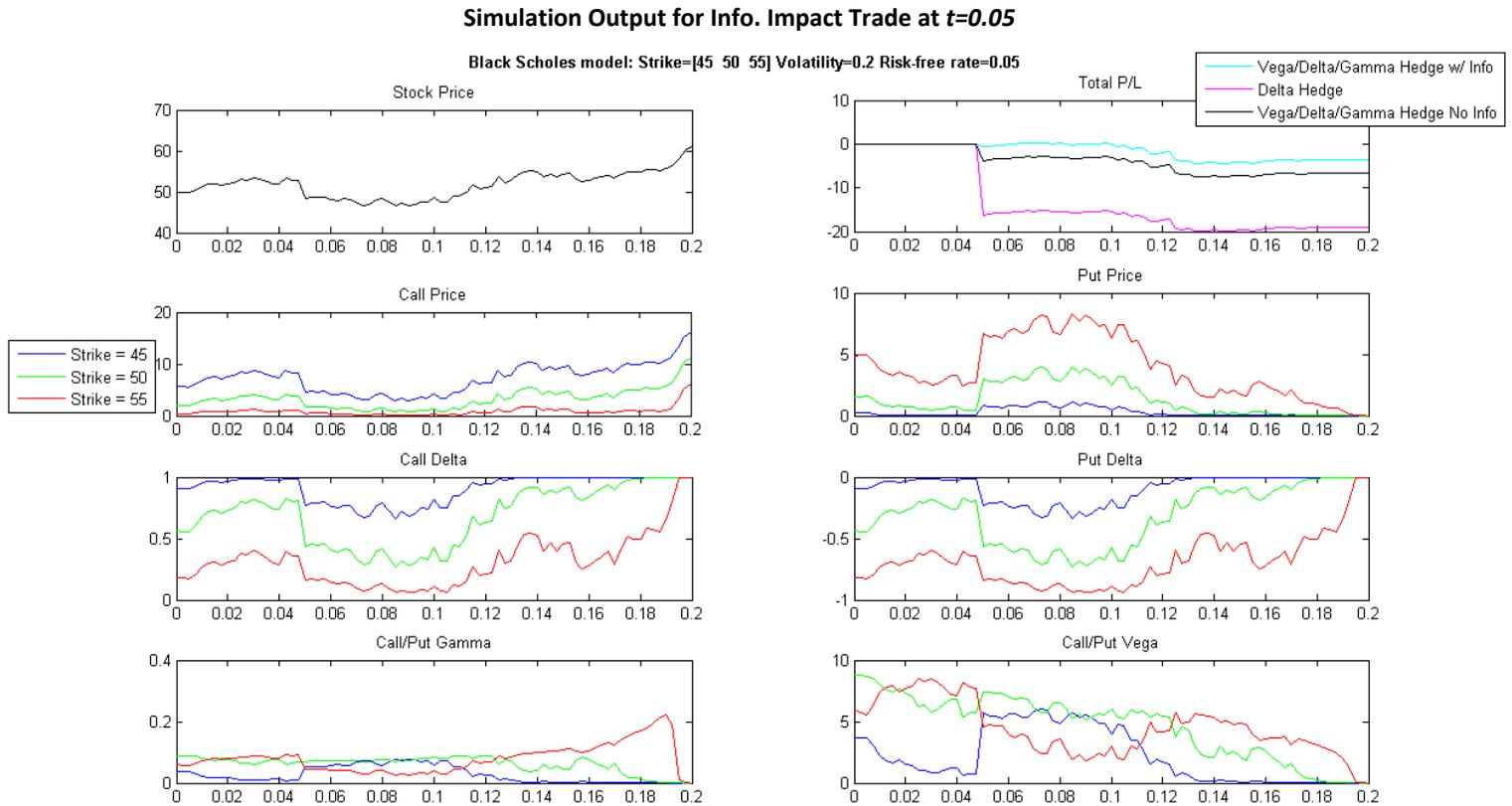
$$\begin{bmatrix} 0.9158 & 0.3578 & -0.0842 & -0.6422 & 1 \\ 0.0258 & 0.0623 & 0.0258 & 0.0623 & 0 \\ 3.0864 & 7.4561 & 3.0864 & 7.4561 & 0 \end{bmatrix} \begin{bmatrix} 5.5025 \\ 7.3236 \\ 0.0001 \\ -0.0001 \\ -0.8769 \end{bmatrix} - \begin{bmatrix} 6.7825 \\ 0.5982 \\ 71.5877 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \Delta \text{ risk} \\ \Gamma \text{ risk} \\ \nu \text{ risk} \end{bmatrix}$$

**Figure 5.4**

A typical simulation of the Vega/Delta/Gamma hedge involved buying some combination of call options while short selling a few assets and/or put options. Intuitively, this result is consistent with what one would expect to minimize the risk. By buying similar call options, the dealer is naturally cancelling out some of the risks that came with selling the call options to the customer. In the above example, the put options are essentially not used in the hedge. Depending on the position risk during a particular simulation, the put options were used sparingly to balance the minimization of the delta, gamma and vega risk. Finally, the asset was sold short in this example in order to fine-tune the delta risk for the position.

The system simulated the proposed hedging strategy and compared the results to two alternative hedging methods. The first alternative is the pure delta hedging strategy (labeled “Delta Hedge” in **Figure 5.5**). The second alternative is a strategy that hedges vega, gamma and delta risk using the same constrained minimization problem defined in equation (32) but has a minor difference (labeled “Vega/Delta/Gamma Hedge No Info” in **Figure 5.5**). The difference is that this alternative does not use the “information” contained in the trade (i.e., this strategy

calculates the risks of the options using the current 0.2 volatility). In other words, this second alternative is the result from a naïve vega, gamma and delta risk hedge. **Figure 5.5** contains plots of the data from one simulation, where the p/l labeled “Vega/Delta/Gamma Hedge w/ Info” is the p/l obtained from using information about the increase in volatility when determining option risks.



**Figure 5.5**

The plot in the top right corner shows the results of the three hedging strategies from a single simulation. A visual inspection shows that the “Delta Hedge” strategy performs the worst. This is expected because this strategy completely ignores the gamma and vega risks. Thus, when the volatility increases, the dealer’s position of short call options will increase by a small amount since the asset price falls. However, as a part of the hedge the trader will have bought some shares of the underlying asset. These shares lose more value than the trader gains from the increase in the value of his/her option position. Analyzing the “Vega/Delta/Gamma Hedge No Info” and the “Vega/Delta/Gamma Hedge w/ Info” require a more detailed statistical analysis that I performed by running the simulation multiple times.

To determine the effectiveness of the each strategy the system ran the randomly generated simulations 1,400 times and determined the p/l for each of the hedging strategies given the outcome of the simulation. In addition to recording the final p/l value the system also recorded:

- The sum of the absolute value of the number of options and assets bought or sold to initiate the hedge.
- The maximum hedge coefficient value  $\min x_i$ .
- The minimum hedge coefficient value  $\max x_i$ .

**Figure 5.6** summarizes the results from these simulations. On average, the Vega/Delta/Gamma hedge that did not incorporate the new volatility value performed the best in terms of total p/l. However, the variance of the p/l distribution was the largest for this hedging strategy as well. In contrast, the average total p/l for the Vega/Delta/Gamma hedge that did incorporate the new volatility value was slightly negative. However, the variance of the p/l distribution for this strategy was the smallest of the three.

**Information Impact Hedge Statistical Results**

Hedge Type	Average P/L	P/L Distribution Variance	Sum of $ x_i $	Average Max Hedge Coefficient	Average Min hedge Coefficient
Delta Hedge	-11.63227522	15.08657411	N/A	N/A	N/A
Vega/Delta/Gamma No Info.	2.376848986	36.34710501	18.3030385	8.797099786	-1.04103963
Vega/Delta/Gamma w/ Info.	-0.7505194	6.09680273	13.52183229	8.065033929	-0.686124093

**Figure 5.6**

The plot in **Figure 5.7** shows the p/l distributions for each of the three hedging strategies. As one would expect the p/l from the delta hedging strategy performed poorly due to the fact that it does not take into account all of the risks associated with the position. As mentioned above it is clear from the distribution that the hedging strategy which takes into account the informational impact of the customer's trade does the best at minimizing risk.

### Distribution of P/L from Hedging Strategies

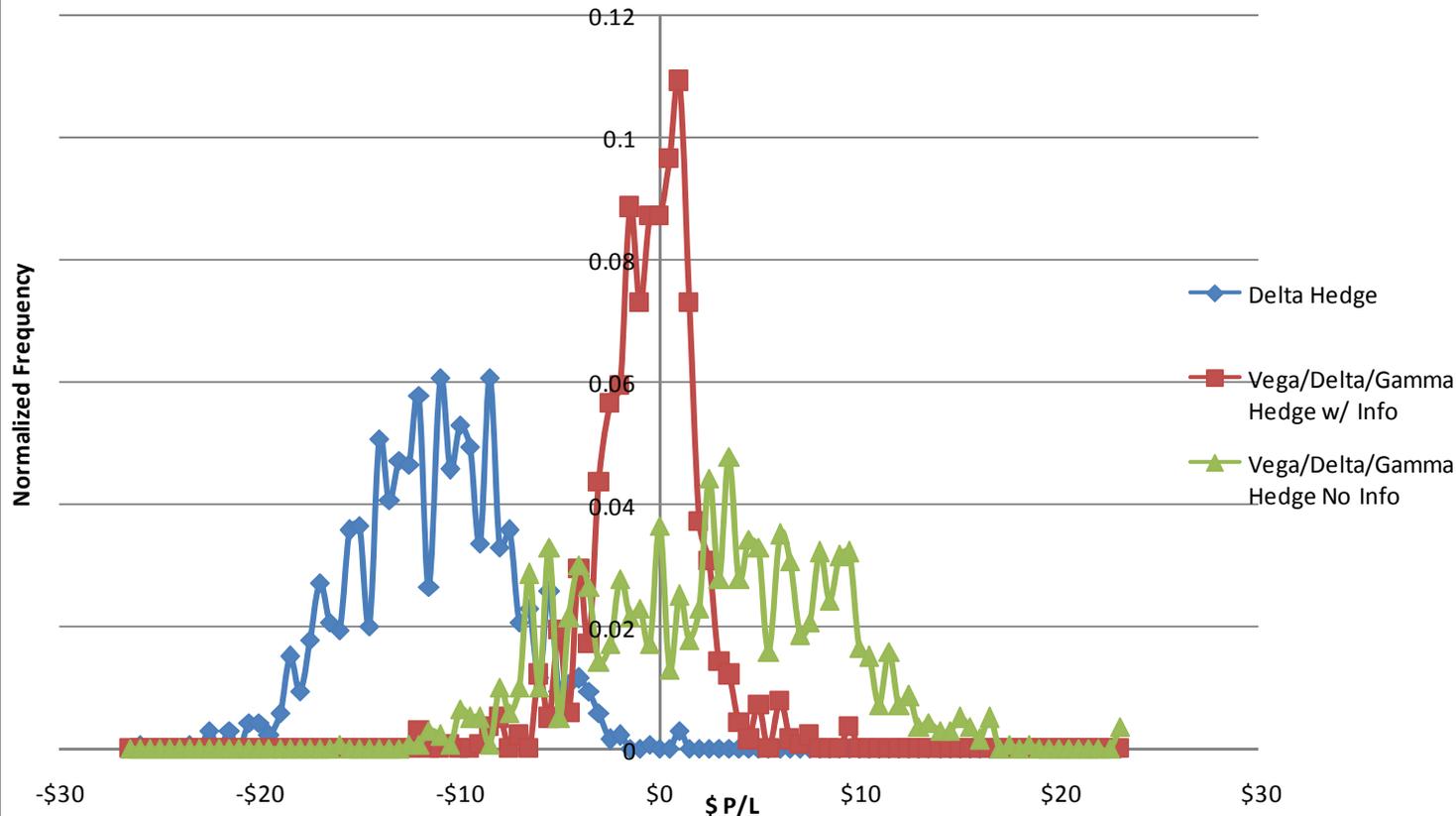


Figure 5.7

The sum of the absolute value of the hedge coefficients was also used to evaluate each hedging strategy. This value measures how many options or assets the dealer would have to buy or sell to initiate the hedge. All else being equal, one would want to buy or sell the fewest number of options to initiate a hedge in order to reduce transactions costs and because of liquidity constraints. Based on the results from the simulations, the hedging strategy that uses the increased volatility value uses the least number of other options and assets to hedge.

The final two metrics recorded by the simulation were the minimum and maximum hedge coefficient values. The maximum value represents the maximum number of any single option or asset bought to initiate the hedge. In real options markets, relying on a single option to hedge a position could be unwise. For instance, volatilities of options are not constant over different strike prices and thus changes in these volatilities might not necessarily move in proportion to each other. Thus, using many options with varying strike prices (both above and below the option position one wishes to hedge) should help average out this discrepancy. For many of the simulations the maximum hedge coefficient was 10 for the hedging strategy that did not use the information from the trade. This result is one explanation for why this hedging strategy had a large variance. On the other hand, the hedging strategy that used the increased volatility value

relied less on an individual option or asset to hedge. Therefore the p/l distribution from this strategy was centered more closely to the mean.

The minimum hedge coefficient can be either a positive or a negative value. If it is negative, then it represents the largest number of options or assets sold to hedge. Otherwise, (when it is positive) it represents the smallest number of options or assets bought to implement the hedge. Since the dynamic hedging strategies mostly buy call options to hedge the position risk the minimum hedge coefficient is usually only slightly negative. It is unclear whether the fact that the “No Info.” hedging strategy has a larger minimum hedge coefficient (in absolute value) is indicative of its worse performance.

## 6. Understanding Risk and Managing Risk

In the complicated and competitive world of options markets companies pursue the cutting edge of innovations in technology, quantitative finance and both human and computer decision making under uncertainty. With all of the advancement over the last few decades, the foundational principle of managing risk has been at the core of the successful operation of financial firms. Unfortunately, throughout the past year, the “gambler’s ruin” is a fate that has become true for many financial institutions who have suffered from a bear market set off by a meltdown of the subprime mortgage market in the United States. In light of these events it has become quite clear that understanding risk and properly managing risk are in many cases two distinct problems.

For market making and broker-dealer firms in particular, much of the risk management is done on a micro level by the individual traders that trade for a company. The ability of these individual traders to optimize their risk is essential for the ongoing success of the firm. The first analysis in this project examined the results of a delta hedging strategy under different Black-Scholes model parameters. Even though the Black-Scholes model does not hold in practice, this analysis can still be used as a benchmark for traders who want to know how different financial environments will react to a delta hedging strategy. Understanding how a delta hedging strategy performs under different scenarios allows a trader to determine how frequently he/she should re-hedge a position.

The analysis of the gambler’s ruin problem showed that there is a tradeoff that exists between taking some risk and surviving as a firm in the future. This tradeoff is dependent on, among other aspects, the firm’s advantages which could be technological, information driven or based on trader experience. The investigation of a block order option trade gave some insight into how a trader can use privileged information in order to execute a more accurate hedging strategy. In particular, the simulations showed that using “information” contained in a trade allows a trader to modify his/her calculation of the risks involved with a trade. Having some idea of what the future risks of a position will be allows a trader to hedge more effectively. Of course, an implicit assumption in this analysis is that the trader understands how to interpret information in a trade. Knowing how to interpret this information is often times a result of experience on behalf of the trader.

At its core, risk is uncertainty. Uncertainty in financial markets is the source (or maybe the cause) of intense competition between individuals seeking profit in a capitalist system. Yet, risk is not always welcome. Market participants have different levels of risk tolerance. The fact that individuals and firms have the ability to manipulate their exposure to risk by entering into contracts with counterparties is truly a marvel of modern financial systems.

## Appendix A: Matlab Code for Delta Hedging Strategy

### A.1: Black-Scholes Formula

```
% input - required inputs for Black-Scholes valuation of an option
% output - vector of option values and deltas
function [output]=blackscholes(input)
stockprice = input(1);
time = input(2);
volatility = input(3);
riskfree = input(4);
strike = input(5);
exptime = input(6);
d1 = (log(stockprice/strike)+(riskfree+.5*volatility^2)*(exptime-time))/(volatility*sqrt(exptime-time));
d2 = d1 - volatility*sqrt(exptime-time);
callprice = stockprice*0.5*erfc(-d1/sqrt(2))-strike*exp(-riskfree*(exptime-time))*0.5*erfc(-d2/sqrt(2));
putprice = strike*exp(-riskfree*(exptime-time))*0.5*erfc(d2/sqrt(2))-stockprice*0.5*erfc(d1/sqrt(2));
calldelta = 0.5*erfc(-d1/sqrt(2));
putdelta = calldelta - 1;
output = [callprice,putprice,calldelta,putdelta];
```

### A.2: Stock Price Generator

```
% generates a vector of stock prices that correspond to the BS model requirements
function [prices]=stockpricegenerator(input)
initialprice = input(1);
length = input(2);
timestep = input(3);
rf = input(4);
volatility = input(5);
prices = zeros(length,2);
prices(1,2) = initialprice;
prices(1,1) = 0.0;
for i = 2:length
    r = randn;
    prices(i,2)=prices(i-1,2)+volatility*prices(i-1,2)*sqrt(timestep)*randn;
    prices(i,1)=prices(i-1,1)+timestep;
end
```

### A.3: Model Simulator

```
% simulates the BS model
global S0 volatility rf strike expirationtime seed r calldelta stockprice callprice stockprice
clc
clear figures
clc
S0 = 50;
volatility = 0.2;
rf = 0.05;
timestep = .005;
strike = 50;
expirationtime = 0.2;
stockprice = stockpricegenerator([S0 expirationtime/timestep+1 timestep rf volatility]);
save('stockprice.mat','stockprice')
```

```

sim('BSHedge');
save('callprice.mat','callprice')
save('calldelta.mat','calldelta')
save('putprice.mat','putprice')
save('putdelta.mat','putdelta')
hedge;
plotresults;

```

#### A.4: Hedge P/L Calculation

% takes price and delta information and produces vectors

% for the call p/l

```
len = length(stockprice);
```

```
callpl = zeros(len,1);
```

```
callpl(1) = 0;
```

```
for t = 2:len
```

```
    callpl(t)=exp(-rf*timestep*(t-1))*(callprice(t)-callprice(t-1)-calldelta(t-1)*(stockprice(2,t)-stockprice(2,t-1))-
(exp(rf*timestep)-1)*(callprice(t-1)-calldelta(t-1)*stockprice(2,t-1)));
```

```
end
```

```
accumcallpl = zeros(len,1);
```

```
accumcallpl(1) = 0;
```

```
for t = 2:len
```

```
    accumcallpl(t) = sum(callpl(1:t));
```

```
end
```

```
totcallpl = accumcallpl(len);
```

## Appendix B: Matlab Code for Information Impact Hedging

### B.1: Black-Scholes Equation (with Multiple Input Strike Prices)

```
% input - required inputs for Black-Scholes valuation of an option
%   with the strike prices as a vector
% output - vector of option values, deltas, vega and gamma
function [output]=blackscholes(input)
stockprice = input(1);
time = input(2);
volatility = input(3);
riskfree = input(4);
exptime = input(5);
strike = input(6:length(input));
for i = 1:length(strike)
    d1 = (log(stockprice/strike(i))+(riskfree+.5*volatility^2)*(exptime-time))/(volatility*sqrt(exptime-time));
    d2 = d1 - volatility*sqrt(exptime-time);
    callprice = stockprice*0.5*erfc(-d1/sqrt(2))-strike(i)*exp(-riskfree*(exptime-time))*0.5*erfc(-d2/sqrt(2));
    putprice = strike(i)*exp(-riskfree*(exptime-time))*0.5*erfc(d2/sqrt(2))-stockprice*0.5*erfc(d1/sqrt(2));
    calldelta = 0.5*erfc(-d1/sqrt(2));
    putdelta = calldelta - 1;
    vega = stockprice*(exp(-d1^2/2)/sqrt(2*pi))*sqrt(exptime-time);
    gamma = exp(-d1^2/2)/(sqrt(2*pi*(exptime-time))*stockprice*volatility);
    if isnan(gamma)
        gamma = 0;
    end
    output(i,:) = [callprice,putprice,calldelta,putdelta,vega,gamma];
end
```

### B.2: Modified Asset Price Generator

```
% generates a vector of stock prices that correspond to the modified
% Black-Scholes framework
% A volatility increase is included at timetoincreasevol
% and a randomly generated (uniformly distributed) asset price
% increase/decrease of 5-10% is included at timetoincreasevol.
function [prices]=infostockpricegenerator(input)
initialprice = input(1);
length = input(2);
timestep = input(3);
rf = input(4);
volatility = input(5);
increasevol = input(6);
timetoincreasevol = input(7);
prices = zeros(length,2);
prices(1,2) = initialprice;
prices(1,1) = 0.0;
prices(2,1) = timestep;
for i = 2:length
    r = randn;
    time = prices(i-1,1)+timestep;
    if time>timetoincreasevol
        if time<=timetoincreasevol+timestep
            if rand < 0.5
```

```

        prices(i,2) = (1.1-1.05)*prices(i-1,2)*rand+(prices(i-1,2)*1.05);
    else
        prices(i,2) = -(1.1-1.05)*prices(i-1,2)*rand+(prices(i-1,2)*.95);
    end
else
    prices(i,2)=prices(i-1,2)+(volatility+increasevol)*prices(i-1,2)*sqrt(timestep)*randn;
end
else
    prices(i,2)=prices(i-1,2)+volatility*prices(i-1,2)*sqrt(timestep)*randn;
end
prices(i,1)=time;
end
end

```

### B.3: Hedge P/L Calculator for Information Impact Trade

```

% takes price and risk information and produces vectors
% for the p/l using the three specified hedging strategies
len = length(stockprice);
hedgepl = zeros(len,1);
noinfohedgepl = zeros(len,1);
deltahedge = zeros(len,1);
hedgepl(1) = 0;
noinfohedgepl(1) = 0;
deltahedge(1) = 0;
size = 10;
for t = 2:len
    if timestep*(t-1) < timetoincreasevol
        deltahedge(t) = 0;
        noinfohedgepl(t) = 0;
        hedgepl(t) = 0;
    else
        % compute the delta hedge p/l for this time step
        deltahedge(t)=size*(-exp(-rf*timestep*(t-1))*(callprice(t,2)-callprice(t-1,2)-calldelta(t-1,2)*(stockprice(2,t)-stockprice(2,t-1))-(exp(rf*timestep)-1)*(callprice(t-1,2)-calldelta(t-1,2)*stockprice(2,t-1)))));
        if timestep*(t-1) ~= timetoincreasevol
            % this is the time step of the info trade
            % calculate the deltas, gammas, vegas of the options in the previous time
            % step knowing that the increase in vol is during this time step
            a = 0;
            b = 0;
            c = 0;
            d = 0;
            e = size*calldelta(t-1,2);
            ani = 0;
            bni = 0;
            cni = 0;
            dni = 0;
            eni = size*calldelta(t-1,2);
        else
            input = [stockprice(2,t-1),timestep*t,volatility+volincrease,rf,expirationtime];
            modeldelta = [Delta([input,strike(1),0]) Delta([input,strike(3),0]) Delta([input,strike(1),1])
Delta([input,strike(3),1]) 1];
            modgamma = [Gamma([input,strike(1)]) Gamma([input,strike(3)]) Gamma([input,strike(1)])
Gamma([input,strike(3)]) 0];
        end
    end
end

```

```

    modvega = [Vega([input,strike(1)]) Vega([input,strike(3)]) Vega([input,strike(1)])
Vega([input,strike(3)]) 0];
    risk = [moddelta;modgamma;modvega]
    posrisk = size*[Delta([input,strike(2),0]),Gamma([input,strike(2)]),Vega([input,strike(2)])]'
    x0 = [hedgecoeffs];
    options = optimset('MaxFunEvals',1e5,'MaxIter',1e4);
    cp1 = callprice(t-1,1);
    cp3 = callprice(t-1,3);
    pp1 = putprice(t-1,1);
    pp3 = putprice(t-1,3);
    sp = stockprice(2,t-1);
    fun = @(x)
abs(x(1))+abs(x(2))+abs(x(3))+abs(x(4))+abs(x(5))+abs(cp1*x(1)+cp3*x(2)+pp1*x(3)+pp3*x(4)+sp*x(5));
    hedgecoeffs = fmincon(fun,x0,[],[],[risk],[posrisk],[-size,-size,-size,-size,-
size],[size,size,size,size,size ],[],[options])
    a = hedgecoeffs(1);
    b = hedgecoeffs(2);
    c = hedgecoeffs(3);
    d = hedgecoeffs(4);
    e = hedgecoeffs(5);
    sumhedgecoeffs = sum(abs(hedgecoeffs));
    maxhedgecoeff = max(hedgecoeffs);
    minhedgecoeff = min(hedgecoeffs);

```

```

    %hedge with no knowledge of info of increased vol
    inputnoinfo = [stockprice(2,t-1),timestep*t,volatility,rf,expirationtime];
    noinfodelta = [Delta([inputnoinfo,strike(1),0]) Delta([inputnoinfo,strike(3),0])
Delta([inputnoinfo,strike(1),1]) Delta([inputnoinfo,strike(3),1]) 1];
    noinfogamma = [Gamma([inputnoinfo,strike(1)]) Gamma([inputnoinfo,strike(3)])]
Gamma([inputnoinfo,strike(1)]) Gamma([inputnoinfo,strike(3)]) 0];
    noinfovega = [Vega([inputnoinfo,strike(1)]) Vega([inputnoinfo,strike(3)])]
Vega([inputnoinfo,strike(1)]) Vega([inputnoinfo,strike(3)]) 0];
    noinfoforisk = [noinfodelta;noinfogaamma;noinfovega]
    noinfoforisk =
size*[Delta([inputnoinfo,strike(2),0]),Gamma([inputnoinfo,strike(2)]),Vega([inputnoinfo,strike(2)])]'
    noinfohedgecoeffs = fmincon(fun,x0,[],[],[noinforisk],[noinfoposrisk],[-size,-size,-size,-size,-
size],[size,size,size,size,size ],[],[options])
    ani = noinfohedgecoeffs(1);
    bni = noinfohedgecoeffs(2);
    cni = noinfohedgecoeffs(3);
    dni = noinfohedgecoeffs(4);
    eni = noinfohedgecoeffs(5);
    sumnoinfohedgecoeffs = sum(abs(noinfohedgecoeffs));
    maxnoinfohedgecoeff = max(noinfohedgecoeffs);
    minnoinfohedgecoeff = min(noinfohedgecoeffs);

```

end

```

hedgepl(t)=exp(-rf*timestep*(t-1))*(...
+a*(callprice(t,1)-callprice(t-1,1))...
-size*(callprice(t,2)-callprice(t-1,2))...
+b*(callprice(t,3)-callprice(t-1,3))...
+c*(putprice(t,1)-putprice(t-1,1))...
+d*(putprice(t,3)-putprice(t-1,3))...
+e*(stockprice(2,t)-stockprice(2,t-1))...
+(exp(rf*timestep)-1)*(size*callprice(t-1,2)-a*callprice(t-1,1)-b*callprice(t-1,3))...

```

```

-c*putprice(t-1,1)-d*putprice(t-1,3)-e*stockprice(2,t-1));

noinfohedgepl(t)=exp(-rf*timestep*(t-1))*(...
+ani*(callprice(t,1)-callprice(t-1,1))...
-size*(callprice(t,2)-callprice(t-1,2))...
+bni*(callprice(t,3)-callprice(t-1,3))...
+cni*(putprice(t,1)-putprice(t-1,1))...
+dni*(putprice(t,3)-putprice(t-1,3))...
+eni*(stockprice(2,t)-stockprice(2,t-1))...
+(exp(rf*timestep)-1)*(size*callprice(t-1,2)-ani*callprice(t-1,1)-bni*callprice(t-1,3)...
-cni*putprice(t-1,1)-dni*putprice(t-1,3)-eni*stockprice(2,t-1)));
end
end

% accumulate the individual time step p/l values
accumhedgepl = zeros(len,1);
accumdeltahedge = zeros(len,1);
accumnoinfohedgepl = zeros(len,1);
accumhedgepl(1) = 0;
accumdeltahedge(1) = 0;
accumnoinfohedgepl(1) = 0;
for t = 2:len
    accumhedgepl(t) = sum(hedgepl(1:t));
    accumdeltahedge(t) = sum(deltahedge(1:t));
    accumnoinfohedgepl(t) = sum(noinfohedgepl(1:t));
end
tothedgepl = accumhedgepl(len);
totaldeltahedge = accumdeltahedge(len);
totnoinfohedgepl = accumnoinfohedgepl(len);

```

## References

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<sup>i</sup> McKernan, James. Gamblers Ruin. < [http://www.math.ucsb.edu/~mckernan/Teaching/06-07/Winter/CCS/l\\_11.pdf](http://www.math.ucsb.edu/~mckernan/Teaching/06-07/Winter/CCS/l_11.pdf)>.

<sup>ii</sup> Black, F. and Scholes, M. S. “The Pricing of Options and Corporate Liabilities.” *J. Political Economy*. 81, 637-659, 1973.

<sup>iii</sup> Wilmott, Paul, Sam Howison, and Jeff Dewynne. The Mathematics of Financial Derivatives: A Student Introduction. Cambridge, UK: Cambridge University Press, 1995.

<sup>iv</sup> Ibid.

<sup>v</sup> Ibid.

<sup>vi</sup> Ibid.

<sup>vii</sup> Ibid.

<sup>viii</sup> Ibid.

<sup>ix</sup> Ibid.

<sup>x</sup> Ibid.

<sup>xi</sup> Ibid.

<sup>xii</sup> Ibid.

<sup>xiii</sup> Call option – WilmottWiki. 21 May 2007. < <http://www.wilmottwiki.com/wiki/index.php/Call>>.

<sup>xiv</sup> Ibid.